A CONVERSE TO LINEAR INDEPENDENCE CRITERIA, VALID ALMOST EVERYWHERE

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Abstract. We prove a weighted analogue of the Khintchine–Groshev Theorem, where the distance to the nearest integer is replaced by the absolute value. This is subsequently applied to proving the optimality of several linear independence criteria over the field of rational numbers.

1. Introduction

Let $n \geq 1$ and let $\psi_1, \ldots, \psi_n : \mathbb{R}^+ \to \mathbb{R}^+$ be functions tending to zero. We will refer to these functions as approximating functions or error functions. Let $\underline{\psi} = (\psi_1, \ldots, \psi_n)$. An $m \times n$-matrix $X = (x_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \in \mathbb{R}^{mn}$ (or a system of linear forms) is said to be $\underline{\psi}$-approximable if

$$|q_1x_{1i} + \cdots + q_mx_{mi}| < \psi_i(|\mathbf{q}|), \quad 1 \leq i \leq n \tag{1}$$

for infinitely many integer vectors $\mathbf{q} = (q_1, \ldots, q_m) \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. The norm $|\mathbf{q}|$ is the supremum norm here and elsewhere. We will denote the set of $\underline{\psi}$-approximable linear forms inside the set $[-\frac{1}{2}, \frac{1}{2}]^{mn}$ by $W_0(m, n, \underline{\psi})$.

The similarity between the $\underline{\psi}$-approximable linear forms studied here and the simultaneously ψ-approximable linear forms usually studied in Diophantine approximation is clear. However, in the classical setup one studies the distance to the nearest integer rather than the absolute value.

A major breakthrough in the classical theory was the Khintchine–Groshev theorem [13, 17], which establishes a zero-one law for the set of $\underline{\psi}$-approximable matrices depending on the convergence or divergence of a certain series. In the absolute value setting, an analogue of this result was recently obtained by Hussain and Levesley [15]. Their result covers only the case $\psi_1 = \cdots = \psi_n$ with this approximating function being monotonic. The condition of monotonicity was removed by Hussain and Kristensen [14] in the case of a single approximating function.

In the present paper, we extend the results of [15] and [14] to the weighted setup, *i.e.*, the case of more than one approximating function. This has applications to linear independence criteria, as we shall see below. Our zero-one law states the following.

Theorem 1.1. *Let $m > n > 0$ and let $\psi_1, \ldots, \psi_n$ be approximating functions as above. Then, if $(m, n) \neq (2, 1)$,*

$$\lambda_{mn}(W_0(m, n, \underline{\psi})) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi_1(r)\cdots\psi_n(r) r^{m-n-1} < \infty \\ 1 & \text{if } \sum_{r=1}^{\infty} \psi_1(r)\cdots\psi_n(r) r^{m-n-1} = \infty, \end{cases}$$

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where λ_{mn} denotes the mn-dimensional Lebesgue measure. If $(m,n) = (2,1)$, the same conclusion holds provided the error function is monotonic.

The case $m \leq n$ is of less interest in general, and of no particular interest to us for applications. Briefly, in this case the set $W_0(m,n,\underline{\psi})$ becomes a subset of a lower dimensional set. An easy instance is that of $m = n = 1$, where it is straightforward to prove that the set is in fact a singleton – see, *e.g.*, Lemma 1 in [6] for details. This is in contrast to the classical case, where approximation to the nearest integer is considered. Here, the result is independent of the relative sizes of m and n.

This setting where linear forms are very small at some points appears in linear independence criteria. To begin with, let us consider the case of one point. Siegel has proved, using essentially a determinant argument, that the existence of m linearly independent linear forms very small at a given point $\mathbf{e}_1 = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ implies a lower bound on the dimension of the $\mathbb{Q}$-vector space spanned by $\xi_1, \ldots, \xi_m$. A precise statement is given by Theorem 1.2 below with assumption (i) and $n = 1$; notice that $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_1, \ldots, \xi_m)$ is equal to the dimension of the smallest subspace F of $\mathbb{R}^m$, defined over the rationals, which contains the point $\mathbf{e}_1 = (\xi_1, \ldots, \xi_m)$. The reader may refer to §8 of [2] for classical facts about subspaces defined over the rationals, to Lemma 1 of [10] (§2.3) for a generalization of this equality, and to [9] (especially pp. 81–82 and 215–216) for more details on Siegel's criterion, including applications.

On the other hand, still in the case of one point $\mathbf{e}_1 = (\xi_1, \ldots, \xi_m)$, Nesterenko has derived [18] a similar lower bound for $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_1, \ldots, \xi_m)$ from the existence of just *one* linear form (for each Q sufficiently large), small at $\mathbf{e}_1$ but not too small: see Theorem 1.2 below with assumption (ii) and $n = 1$. The most striking application of his result is the proof by Rivoal [19] and Ball-Rivoal [1] that infinitely many values of Riemann ζ function at odd integers $s \geq 3$ are irrational.

The first author has generalized recently Nesterenko's linear independence criterion to linear forms small at several points (see [10], Theorem 3). The statement is the following, with assumption (ii). We provide also (under assumption (i)) the analogue of Siegel's criterion in this setting (see [10], §2.4, Proposition 1). We denote by $\cdot$ the canonical scalar product on $\mathbb{R}^m$ (which allows us to consider a linear form as the scalar product with a given vector), and by $o(1)$ any sequence that tends to 0 as $Q \to \infty$.

Theorem 1.2. *Let $m > n > 0$, and $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^m$. Let $\tau_1, \ldots, \tau_n$ be positive real numbers. Assume that one of the following holds:*

- (i) *The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent, and for infinitely many integers Q there exist m linearly independent vectors $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)} \in \mathbb{Z}^m$ such that, for any $j \in \{1, \ldots, m\}$:*
$$|\mathbf{q}^{(j)}| \leq Q \text{ and } |\mathbf{q}^{(j)} \cdot \mathbf{e}_i| \leq Q^{-\tau_i + o(1)} \text{ for any } i \in \{1, \ldots, n\}.$$
- (ii) *The numbers $\tau_1, \ldots, \tau_n$ are pairwise distinct, and for any sufficiently large integer Q there exists $\mathbf{q} \in \mathbb{Z}^m$ such that*
$$|\mathbf{q}| \leq Q \text{ and } |\mathbf{q} \cdot \mathbf{e}_i| = Q^{-\tau_i + o(1)} \text{ for any } i \in \{1, \ldots, n\}.$$

Then we have
$$\dim F \geq n + \tau_1 + \ldots + \tau_n$$
for any subspace F of $\mathbb{R}^m$ which contains $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and is defined over the rationals.

Note that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are always $\mathbb{R}$-linearly independent: this is assumed in (i), and it is an easy consequence of assumption (ii) since $\tau_1, \ldots, \tau_n$ are pairwise distinct (see [10], §3.2). The point is that $\mathrm{Span}_{\mathbb{R}}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is not defined over the rationals.

The conclusion of Theorem 1.2 is a lower bound for $\dim F$ (which can be stated as a lower bound for the rank of a family of m vectors in $\mathbb{R}^n$ seen as a $\mathbb{Q}$-vector space, see [10], §2.3, Lemma 1). It is a natural question to ask whether this bound can be improved; we give a negative answer in Theorem 1.3. In the case of Nesterenko's linear independence criterion with only one point, Chantanasiri has given ([5], §3) a very specific example of a point $\mathbf{e}_1 = (\xi_1, \ldots, \xi_m)$ for which this bound is optimal (namely when $(\xi_1, \ldots, \xi_m)$ is a $\mathbb{Q}$-basis of a real number field of degree m). On the contrary, our result deals with generic tuples; it encompasses also Siegel's criterion, and the case of several points.

Theorem 1.3. *Let $m > n > 0$, and F be a subspace of $\mathbb{R}^m$ defined over the rationals. Let $\tau_1, \ldots, \tau_n$, $\beta_1, \ldots, \beta_n$, ε be real numbers such that $\tau_1 > 0$, $\ldots$, $\tau_n > 0$, $\varepsilon > 0$,*

$$\tau_1 + \ldots + \tau_n \leq \dim F - n \text{ and } \beta_1 + \ldots + \beta_n = (1+\varepsilon)(\dim F - 1). \tag{2}$$

Then for almost all n-tuples $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in F^n$ (with respect to Lebesgue measure) the following property holds. For any sufficiently large integer Q there exist m linearly independent vectors $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)} \in \mathbb{Z}^m$ such that, for any $j \in \{1, \ldots, m\}$:

$$|\mathbf{q}^{(j)}| \ll Q \tag{3}$$

and

$$Q^{-\tau_i} (\log Q)^{\beta_i - (1+\varepsilon)\dim F} \ll |\mathbf{q}^{(j)} \cdot \mathbf{e}_i| \ll Q^{-\tau_i} (\log Q)^{\beta_i} \text{ for any } i \in \{1, \ldots, n\}, \tag{4}$$

where the constants implied in the symbols $\ll$ depend on m, n, F, $\tau_1, \ldots, \tau_n$, $\beta_1, \ldots, \beta_n$, ε, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ but not on Q.

This result will be proved in §2.2, using Theorem 1.1 and Minkowski's theorem on successive minima of a convex body. We also postpone until §2.1 some remarks on Theorem 1.3.

Throughout we will use the Vinogradov notation, *i.e.*, for two real quantities x and y, we will write $x \ll y$ if there is a constant $C > 0$ such that $x \leq Cy$. In Landau's O-notation this would amount to writing $x = O(y)$. If $x \ll y$ and $y \ll x$, we will write $x \asymp y$.

2. A converse to linear independence criteria

2.1. **Remarks on Theorem 1.3.** We gather in this section several remarks on Theorem 1.3.

Remark 1. In general Nesterenko's criterion is stated under a slightly different assumption than (*ii*) in Theorem 1.2: it is assumed that there exist an increasing sequence $(Q_k)_{k\geq 1}$ of positive integers such that $Q_{k+1} = Q_k^{1+o(1)}$ as $k \to \infty$ (where the sequence denoted by $o(1)$ tends to 0 as $k \to \infty$), and a sequence $(\mathbf{q}_k)_{k\geq 1}$ of vectors in $\mathbb{Z}^m$, such that for any k:

$$|\mathbf{q}_k| \leq Q_k \text{ and } |\mathbf{q}_k \cdot \mathbf{e}_i| = Q_k^{-\tau_i + o(1)} \text{ for any } i \in \{1, \ldots, n\}.$$

Requesting also $\tau_1, \ldots, \tau_n$ to be pairwise distinct, this is actually equivalent to assumption (*ii*) of Theorem 1.2. In precise terms, if there is such a sequence (Q_k) then for any Q sufficiently large one may choose the integer k such that $Q_k \leq Q < Q_{k+1}$, and let $\mathbf{q} = \mathbf{q}_k$. The converse is easy too: if assumption (*ii*) of Theorem 1.2 holds, then one can choose *any* increasing sequence $(Q_k)_{k\geq 1}$ of positive integers such that

$Q_{k+1} = Q_k^{1+o(1)}$ (for instance $Q_k = \beta^k$ with an arbitrary $\beta > 1$) and let $\mathbf{q}_k$ be the vector corresponding to $Q = Q_k$.

This remark shows that $\tau_r(\underline{\xi}) = \tau_r'(\underline{\xi}) = \tau_r''(\underline{\xi})$ for any $\underline{\xi}$ in the notation of §4.3 of [11]. With the same notation, Theorem 1.3 (with $F = \mathbb{R}^m$ and $n = 1$) implies that this Diophantine exponent is equal to $m - 1$ for almost all $\underline{\xi} = \mathbf{e}_1 \in \mathbb{R}^m$ (with respect to Lebesgue measure); this answers partly a question asked at the end of [11].

Remark 2. In the setting of Theorem 1.3, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are $\mathbb{Q}$-linearly independent and belong to $F \cap \overline{\mathbb{Q}}^m$ then applying Schmidt's Subspace Theorem instead of Theorem 1.1 in the proof yields the same conclusion as that of Theorem 1.3, except that Eq. (4) is weakened to $|\mathbf{q}^{(j)} \cdot \mathbf{e}_i| = Q^{-\tau_i + o(1)}$.

In the rest of this section, we shall focus on the special case $m = 2$, $n = 1$, $F = \mathbb{R}^2$. By homogeneity we may restrict to vectors $\mathbf{e}_1 = (\xi, -1)$ with $\xi \in \mathbb{R}$. Since non-zero linear forms in ξ and -1 with integer coefficients are bounded from below in absolute value if ξ is a rational number, we assume ξ to be irrational. Recall that the irrationality exponent of ξ, denoted by $\mu(\xi)$, is the supremum (possibly $+\infty$) of the set of $\mu > 0$ such that there exist infinitely many $p, q \in \mathbb{Z}$ with $q > 0$ such that $|\xi - \frac{p}{q}| \leq q^{-\mu}$. Then the first question related to Theorem 1.3 is to know for which $\tau > 0$ the following holds:

$$\text{For any } Q \text{ there exists } \mathbf{q} = (q_1, q_2) \in \mathbb{Z}^2 \setminus \{(0,0)\} \text{ such that} \\ |\mathbf{q}| \leq Q \text{ and } |q_1 \xi - q_2| = Q^{-\tau + o(1)}. \tag{5}$$

Lemma 1 and Theorem 2 of [11] imply (using Remark 1 above) that (5) holds if, and only if, $\tau < \frac{1}{\mu(\xi)-1}$ (except maybe for $\tau = \frac{1}{\mu(\xi)-1}$: this case is not settled in [11]). This gives a satisfactory answer for any given ξ, and it would be interesting to generalize it to arbitrary values of m and n: questions in this respect are asked (in the case $n = 1$) in §4 of [11]. This result shows also that the conclusion of Theorem 1.3 does not hold for *any* $\mathbf{e}_1, \ldots, \mathbf{e}_n$: property (5) fails to hold for $\tau = 1$ if $\mu(\xi) > 2$.

If ξ is generic (with respect to Lebesgue measure), then $\mu(\xi) = 2$ and the question left open in [11] is whether property (5) holds for $\tau = 1$. Theorem 1.3 answers this question: it does, and the error term $Q^{o(1)}$ can be bounded between powers of $\log Q$. Moreover, Theorem 1.3 provides, for any Q, two linearly independent vectors $\mathbf{q}$ as in (5): as far as we know, no result in the style of [11] provides this conclusion for a non-generic ξ.

In the same situation (namely with $m = 2$, $n = 1$, $F = \mathbb{R}^2$, and a generic ξ), Theorem 1.3 with $\tau_1 = 1$ and $\beta_1 > 1$ provides (for any Q) two linearly independent vectors $\mathbf{q} = (q_1, q_2) \in \mathbb{Z}^2$ such that $|\mathbf{q}| \ll Q$ and

$$Q^{-1} (\log Q)^{-\beta_1} \ll |q_1 \xi - q_2| \ll Q^{-1} (\log Q)^{\beta_1}. \tag{6}$$

The lower bound on $|q_1 \xi - q_2|$ is natural since for infinitely many Q there exists $\mathbf{q}$ such that $|\mathbf{q}| \leq Q$ and $Q^{-1} (\log Q)^{-\beta_1} \ll |q_1 \xi - q_2| \ll Q^{-1} (\log Q)^{-1}$. The upper bound in Eq. (6) could seem too large, since Dirichlet's pigeonhole principle yields (for any Q) a non-zero $\mathbf{q}$ such that $|\mathbf{q}| \leq Q$ and $|q_1 \xi - q_2| \ll Q^{-1}$. However it is possible (by adapting the proof of Theorem 1.3) to prove that, for infinitely many Q, all vectors $\mathbf{q} \in \mathbb{Z}^2$ such that $|\mathbf{q}| \ll Q$ and $|q_1 \xi - q_2| \ll Q^{-1}$ are collinear. To obtain two linearly independent such vectors, one needs (for infinitely many Q) to let $|q_1 \xi - q_2|$ increase

a little more, at least up to $Q^{-1}\log Q$: the upper bound in Eq. (6) is optimal (except that the case $\beta_1 = 1$ could probably be considered, upon multiplying by a power of $\log\log Q$).

2.2. **Proof of Theorem 1.3.** Before proving Theorem 1.3, let us outline the strategy in the case where $F = \mathbb{R}^m$ and $\tau_1 + \ldots + \tau_n = \dim F - n$ (from which we shall deduce the general case). The convex body $\mathcal{C} \subset \mathbb{R}^m$ defined by (3) and the second inequality in (4) has volume essentially equal to a power of $\log Q$. There are non-zero integer points $\mathbf{q}$ inside $\mathcal{C}$, but not "too far away inside" (for Q sufficiently large) : if $\mathbf{q}$ is such a point and $\mu > 0$ is such that $\mu\mathbf{q} \in \mathcal{C}$, then μ is less than some power of $\log Q$ (otherwise the scalar products $|\mathbf{q}\cdot\mathbf{e}_i|$ would be too small: this would contradict the convergent case of Theorem 1.1). This is a lower bound on the first successive minimum λ_1 of $\mathcal{C}$. Using Minkowski's convex body theorem, this yields an upper bound on the last successive minimum λ_m, namely $\lambda_m \ll 1$. This concludes the proof, except for the lower bound in Eq. (4) for which the argument is similar: if $|\mathbf{q}^{(j)}\cdot\mathbf{e}_i|$ is too small for some i, j then $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is not generic (using again the convergent case of Theorem 1.1).

Let us come now to a detailed proof of Theorem 1.3, starting with the following remark.

Remark 3. The general case of Theorem 1.3 follows from the special case where the inequality in Eq. (2) is an equality, that is $\tau_1+\ldots+\tau_n = \dim F - n$. Indeed in general we have $\tau_1 + \ldots + \tau_n = \eta(\dim F - n)$ with $0 < \eta \le 1$, and applying the special case with τ_1/η, ..., τ_n/η and Q^η yields the desired conclusion.

As a first step, let us assume that Theorem 1.3 holds if $F = \mathbb{R}^m$, and deduce the general case. Since F is defined over $\mathbb{Q}$, there exists a basis $(\mathbf{u}_1, \ldots, \mathbf{u}_d)$ of F consisting in vectors of $\mathbb{Z}^m$ (where $d = \dim F$; notice that Eq. (2) implies $d > n$). Let $\Phi : \mathbb{R}^d \to \mathbb{R}^m$ be the linear map which sends the canonical basis of $\mathbb{R}^d$ to $(\mathbf{u}_1, \ldots, \mathbf{u}_d)$. The special case of Theorem 1.3 applies to $\mathbb{R}^d$ (with the same parameters); it provides a subset $\tilde{A} \subset (\mathbb{R}^d)^n$ of full Lebesgue measure, and for any $(\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n) \in \tilde{A}$ and any Q sufficiently large d linearly independent vectors $\tilde{\mathbf{q}}^{(1)}, \ldots, \tilde{\mathbf{q}}^{(d)} \in \mathbb{Z}^d$. Then we let $A \subset F^n$ denote the set of all n-tuples $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ given by $\mathbf{e}_1 = \Phi(\tilde{\mathbf{e}}_1)$, ..., $\mathbf{e}_n = \Phi(\tilde{\mathbf{e}}_n)$ with $(\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n) \in \tilde{A}$; this subset A has full Lebesgue measure in $F^n = (\operatorname{Im}\Phi)^n$.

Let us denote by $\Omega \in M_d(\mathbb{R})$ the matrix in the basis $(\mathbf{u}_1, \ldots, \mathbf{u}_d)$ of the scalar product of $\mathbb{R}^m$ restricted to F. This means that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we have $\Phi(\mathbf{x})\cdot\Phi(\mathbf{y}) = {}^t\mathbf{x}\Omega\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are seen as column vectors (indeed they are the vectors of coordinates in the basis $(\mathbf{u}_1, \ldots, \mathbf{u}_d)$ of $\Phi(\mathbf{x})$ and $\Phi(\mathbf{y})$ respectively). This matrix Ω has integer coefficients (given by $\mathbf{u}_k\cdot\mathbf{u}_\ell$ for $1 \le k, \ell \le d$), and a non-zero determinant, so that $(\det\Omega)\Omega^{-1}$ is a matrix with integer coefficients.

Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in A$, and Q be sufficiently large. We let

$$\mathbf{q}^{(j)} = \Phi\Big((\det\Omega)\Omega^{-1}\tilde{\mathbf{q}}^{(j)}\Big) \text{ for any } j \in \{1, \ldots, d\},$$

so that

$$\mathbf{q}^{(j)}\cdot\mathbf{e}_i = {}^t\Big((\det\Omega)\Omega^{-1}\tilde{\mathbf{q}}^{(j)}\Big)\Omega\tilde{\mathbf{e}}_i = (\det\Omega)\tilde{\mathbf{q}}^{(j)}\cdot\tilde{\mathbf{e}}_i \text{ for any } i \in \{1, \ldots, n\}$$

because Ω is symmetric. Therefore Eqns. (3) and (4) hold for $j \le d$; moreover $\mathbf{q}^{(1)}$, ..., $\mathbf{q}^{(d)}$ are linearly independent vectors in $\mathbb{Z}\mathbf{u}_1 + \ldots + \mathbb{Z}\mathbf{u}_d \subset F \cap \mathbb{Z}^m$ (because the coefficients of $(\det\Omega)\Omega^{-1}$ are integers).

Since $F^\perp$ is a subspace of $\mathbb{R}^m$ defined over the rationals (because F is), there exists a basis $(\mathbf{v}_{d+1}, \ldots, \mathbf{v}_m)$ of $F^\perp$ consisting in vectors of $\mathbb{Z}^m$. Then we let

$$\mathbf{q}^{(d+1)} = \mathbf{v}_{d+1} + \mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)} = \mathbf{v}_m + \mathbf{q}^{(1)}.$$

Then $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)}$ are linearly independent vectors in $\mathbb{Z}^m$, and for any $j \in \{d+1, \ldots, m\}$ and any $i \in \{1, \ldots, n\}$ we have $\mathbf{q}^{(j)} \cdot \mathbf{e}_i = \mathbf{q}^{(1)} \cdot \mathbf{e}_i$ so that Eq. (4) holds. Since $\mathbf{v}_{d+1}, \ldots, \mathbf{v}_m$ can be chosen independently from Q, we have also $|\mathbf{q}^{(j)}| \ll |\mathbf{q}^{(1)}| \ll Q$ so that Eq. (3) holds too. This concludes the proof that the full generality of Theorem 1.3 follows from the special case where $F = \mathbb{R}^m$.

From now on, we assume that $F = \mathbb{R}^m$ and prove Theorem 1.3 in this case.

Let A_0 denote the set of all $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in (\mathbb{R}^m)^n$ such that the system of inequalities

$$|\mathbf{q} \cdot \mathbf{e}_i| \le |\mathbf{q}|^{-\tau_i} (\log |\mathbf{q}|)^{\beta_i - (1+\varepsilon)(1+\tau_i)} \text{ for any } i \in \{1, \ldots, n\} \tag{7}$$

holds for only finitely many $\mathbf{q} \in \mathbb{Z}^m$.

For any $i_0 \in \{1, \ldots, n\}$, let A_{i_0} denote the set of all $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in (\mathbb{R}^m)^n$ such that the system of inequalities

$$\begin{cases} |\mathbf{q} \cdot \mathbf{e}_i| \le c_1 |\mathbf{q}|^{-\tau_i} (\log |\mathbf{q}|)^{\beta_i} \text{ for any } i \in \{1, \ldots, n\}, i \ne i_0 \\ |\mathbf{q} \cdot \mathbf{e}_{i_0}| \le |\mathbf{q}|^{-\tau_{i_0}} (\log |\mathbf{q}|)^{\beta_{i_0} - m(1+\varepsilon)} \end{cases} \tag{8}$$

holds for only finitely many $\mathbf{q} \in \mathbb{Z}^m$; here c_1 is a positive constant the will be defined later in the proof (namely in Eq. (12)), but could have be made explicit and stated here.

Using Eq. (2) and Remark 3, the convergent case of Theorem 1.1 (with $(x_{1i}, \ldots, x_{mi}) = \mathbf{e}_i$) implies that $A_i \cap [-\frac{1}{2}, \frac{1}{2}]^{nm}$ has full Lebesgue measure for any $i \in \{0, \ldots, n\}$. Since A_i is stable under multiplication by scalars, we have $A_i = \cup_{n \in \mathbb{N}} n (A_i \cap [-\frac{1}{2}, \frac{1}{2}]^{nm})$ so that A_i has full Lebesgue measure. At last, let A_∞ denote the set of all $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in (\mathbb{R}^m)^n$ such that $\mathbf{q} \cdot \mathbf{e}_i \ne 0$ for any $\mathbf{q} \in \mathbb{Z}^m \setminus \{0\}$ and any $i \in \{1, \ldots, n\}$. Then we let $A = A_0 \cap A_1 \cap \ldots \cap A_n \cap A_\infty$, and A has full Lebesgue measure in $(\mathbb{R}^m)^n$.

Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in A$, and Q be sufficiently large. Let $\mathcal{C}$ denote the set of all $\mathbf{q} \in \mathbb{R}^m$ such that

$$|\mathbf{q}| \le Q \text{ and } |\mathbf{q} \cdot \mathbf{e}_i| \le Q^{-\tau_i} (\log Q)^{\beta_i} \text{ for any } i \in \{1, \ldots, n\}. \tag{9}$$

Then $\mathcal{C}$ is convex, compact, and symmetric with respect to the origin. Its volume (denoted by $\mathrm{vol}(\mathcal{C})$) is such that $\mathrm{vol}(\mathcal{C}) \asymp (\log Q)^{(1+\varepsilon)(m-1)}$, using both equalities of Eq. (2) (thanks to Remark 3) with $\dim F = m$.

For any $j \in \{1, \ldots, m\}$ let λ_j denote the infimum of the set of all positive real numbers λ such that $\mathbb{Z}^m \cap \lambda \mathcal{C}$ contains j linearly independent vectors, where $\lambda \mathcal{C} = \{\lambda \mathbf{q}, \, \mathbf{q} \in \mathcal{C}\}$. These λ_j are the successive minima of the convex body $\mathcal{C}$ with respect to the lattice $\mathbb{Z}^m$; Minkowski's theorem (see for instance [4], Chapter VIII) yields $\frac{2^m}{m!} \le \lambda_1 \ldots \lambda_m \mathrm{vol}(\mathcal{C}) \le 2^m$, so that

$$\lambda_1 \ldots \lambda_m \asymp (\log Q)^{-(1+\varepsilon)(m-1)}. \tag{10}$$

Since $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in A_0$, for any $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ there exists $i \in \{1, \ldots, n\}$ (which depends on $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and $\mathbf{q}$) such that

$$|\mathbf{q} \cdot \mathbf{e}_i| \gg |\mathbf{q}|^{-\tau_i} (\log |\mathbf{q}|)^{\beta_i - (1+\varepsilon)(1+\tau_i)} \tag{11}$$

where the constant implied in the symbol $\gg$ is small enough to take into account the finitely many $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ that satisfy Eq. (7); we have used here that $\mathbf{q} \cdot \mathbf{e}_i \ne 0$ for any $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ and any i, because $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in A_\infty$.

Let us deduce from this property that $\lambda_1 \gg (\log Q)^{-(1+\varepsilon)}$. With this aim in view, we let $\lambda > 0$ be such that $Q^{-1/2} \leq \lambda \leq 1$ and $\lambda\mathcal{C} \cap \mathbb{Z}^m \neq \{\mathbf{0}\}$; we are going to prove that $\lambda \gg (\log Q)^{-(1+\varepsilon)}$. There exists $\mathbf{q}' \in \mathcal{C}$ such that $\mathbf{q} = \lambda\mathbf{q}' \in \mathbb{Z}^m$ and $\mathbf{q} \neq \mathbf{0}$. Then Eq. (11) provides an integer $i \in \{1, \ldots, n\}$ such that, using Eq. (9):

$$|\mathbf{q}|^{-\tau_i}(\log|\mathbf{q}|)^{\beta_i-(1+\varepsilon)(1+\tau_i)} \ll |\mathbf{q}\cdot\mathbf{e}_i| = \lambda|\mathbf{q}'\cdot\mathbf{e}_i| \leq \lambda Q^{-\tau_i}(\log Q)^{\beta_i}.$$

Since we have also $|\mathbf{q}| = \lambda|\mathbf{q}'| \leq \lambda Q$ and $Q^{-1/2} \leq \lambda \leq 1$ (so that $\log(\lambda Q) \gg \log Q$), this yields

$$\lambda^{-\tau_i}Q^{-\tau_i}(\log Q)^{\beta_i-(1+\varepsilon)(1+\tau_i)} \ll (\lambda Q)^{-\tau_i}(\log(\lambda Q))^{\beta_i-(1+\varepsilon)(1+\tau_i)} \ll \lambda Q^{-\tau_i}(\log Q)^{\beta_i},$$

thereby proving that $\lambda \gg (\log Q)^{-(1+\varepsilon)}$. This concludes the proof that $\lambda_1 \gg (\log Q)^{-(1+\varepsilon)}$; since $\lambda_1 \leq \ldots \leq \lambda_m$ by definition of the successive minima, this implies $\lambda_j \gg (\log Q)^{-(1+\varepsilon)}$ for any $j \in \{1, \ldots, m\}$. Plugging this lower bound for $j \leq m-1$ into Eq. (10) yields $\lambda_m \ll 1$: there exist linearly independent vectors $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)} \in \mathbb{Z}^m$ such that

$$|\mathbf{q}^{(j)}| \ll Q \text{ and } |\mathbf{q}^{(j)}\cdot\mathbf{e}_i| \ll Q^{-\tau_i}(\log Q)^{\beta_i} \text{ for any } i \in \{1, \ldots, n\}. \tag{12}$$

This concludes the proof of Eq. (3), and that of the upper bound in Eq. (4).

To prove the lower bound in Eq. (4), we start by noticing that Eq. (12) yields

$$|\mathbf{q}^{(j)}\cdot\mathbf{e}_i| \leq c_1 Q^{-\tau_i}(\log Q)^{\beta_i} \text{ for any } i \in \{1, \ldots, n\} \tag{13}$$

for some positive constant c_1 (which could be made explicit); this constant is the one used in the definition of A_{i_0} at the beginning of the proof. Now let $i_0 \in \{1, \ldots, n\}$. Since $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in A_{i_0}$ and $\mathbf{q}\cdot\mathbf{e}_{i_0} \neq 0$ for any $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$, there exists a positive constant c_2 such that no non-zero $\mathbf{q} \in \mathbb{Z}^m$ satisfies the system of inequalities

$$\begin{cases} |\mathbf{q}\cdot\mathbf{e}_i| \leq c_1|\mathbf{q}|^{-\tau_i}(\log|\mathbf{q}|)^{\beta_i} \text{ for any } i \in \{1, \ldots, n\}, i \neq i_0 \\ |\mathbf{q}\cdot\mathbf{e}_{i_0}| \leq c_2|\mathbf{q}|^{-\tau_{i_0}}(\log|\mathbf{q}|)^{\beta_{i_0}-m(1+\varepsilon)} \end{cases} \tag{14}$$

For any $j \in \{1, \ldots, m\}$, the non-zero vector $\mathbf{q}^{(j)}$ satisfies the first family of inequalities in (14) (thanks to Eq. (13)), so that

$$|\mathbf{q}^{(j)}\cdot\mathbf{e}_{i_0}| > c_2|\mathbf{q}^{(j)}|^{-\tau_{i_0}}(\log|\mathbf{q}^{(j)}|)^{\beta_{i_0}-m(1+\varepsilon)} \gg Q^{-\tau_{i_0}}(\log Q|)^{\beta_{i_0}-m(1+\varepsilon)}$$

since $|\mathbf{q}^{(j)}| \ll Q$. This concludes the proof of the lower bound in Eq. (4), and that of Theorem 1.3.

3. Proof of Theorem 1.1

3.1. **Convergence case for any choice of m and n.** In order to prove the convergence case, we will exhibit a family of covers of $W_0(m,n,\underline{\psi})$. The covers will be the natural ones, *i.e.*, the cover of $W_0(m,n,\underline{\psi})$ by the sets of solutions to (1) for each individual $\mathbf{q}$. Denote for each $\mathbf{q} \in \mathbb{Z}^m$, the set of matrices with entries in $[-1/2, 1/2]$ satisfying the system of inequalities (1) by $B(\mathbf{q},\underline{\psi})$. It is straightforward to see that

$$\lambda_{mn}(B(\mathbf{q},\underline{\psi})) \asymp \psi_1(|\mathbf{q}|)\cdots\psi_n(|\mathbf{q}|)\,|\mathbf{q}|^{-n}\,. \tag{15}$$

Here, the implied constants depend on m and n.

Secondly, we will need to estimate the number of $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ of a given norm, r say. This is however easily seen to be at most $(2m-1)r^{m-1}$, and so comparable with r^{m-1}.

We now estimate the Lebesgue measure of $W_0(m,n,\underline{\psi})$ under the assumption of convergence. For each $N \geq 1$,

$$\lambda_{mn}\left(W_0(m,n,\underline{\psi})\right) \leq \lambda_{mn}\left(\bigcup_{r\geq N}\bigcup_{|\mathbf{q}|=r} B(\mathbf{q},\underline{\psi})\right) \leq \sum_{r\geq N}\sum_{|\mathbf{q}|=r}\lambda_{mn}\left(B(\mathbf{q},\underline{\psi})\right)$$
$$\leq \sum_{r\geq N}\sum_{|\mathbf{q}|=r}\psi_1(r)\cdots\psi_n(r)r^{-n} \ll \sum_{r\geq N}\psi_1(r)\cdots\psi_n(r)r^{m-n-1}.$$

We have used (15) and the counting estimates. The final sum is the tail of a convergent series, which tends to zero as N tends to infinity.

3.2. **Divergence case.** We give a general approach to the problem in question which has been adapted from the one used in [14]. In the case $(m,n) \neq (2,1)$, we will not need the assumption of monotonicity of the approximating functions. This will be clear from the proof below.

For each $\mathbf{q} \in \mathbb{Z}^{m-n}$, let

$$B_{\mathbf{q}} = \bigcup_{\substack{\mathbf{p}\in\mathbb{Z}^n\\ |\mathbf{p}|\leq|\mathbf{q}|}} \{A \in M_{m\times n}([-1/2,1/2]) : |(\mathbf{p},\mathbf{q})A|_i \leq \psi_i(|\mathbf{q}|)\}$$

Writing each $A \in M_{m\times n}([-1/2,1/2])$ as $\binom{I_n}{\tilde{A}}X$, where X is the $n\times n$ matrix formed by the first n rows of A, we find the related set

$$B'_{\mathbf{q}}(X) = \bigcup_{\substack{\mathbf{p}\in\mathbb{Z}^n\\ |\mathbf{p}|\leq|\mathbf{q}|}} \left\{\tilde{A} \in M_{(m-n)\times n}([-1/2,1/2]) : \left|pX + \mathbf{q}\tilde{A}X\right|_i \leq \psi_i(|\mathbf{q}|)\right\}.$$

Finally, set $B'_{\mathbf{q}} = B'_{\mathbf{q}}(I_n)$.

Let $\epsilon > 0$ be fixed and sufficiently small. We will be more explicit later. From now on, we restrict ourselves to considering matrices A for which the determinant of the matrix X consisting of the first n rows of A is $> \epsilon$. Evidently, this determinant is also $\leq n!$. This immediately implies that X is invertible with $(n!)^{-1} \leq |\det(X^{-1})| < \epsilon^{-1}$.

Lemma 3.1. *For each $X \in M_n([-1/2,1/2])$ with $|\det(X)| > \epsilon$, and each $\mathbf{q},\mathbf{q}_1,\mathbf{q}_2 \in \mathbb{Z}^{m-n}$,*

$$\lambda_{(m-n)n}(B'_{\mathbf{q}}(X)) \asymp_\epsilon \lambda_{(m-n)n}(B'_{\mathbf{q}}),$$

and

$$\lambda_{(m-n)n}(B'_{\mathbf{q}_1}(X)\cap B'_{\mathbf{q}_2}(X)) \asymp_\epsilon \lambda_{(m-n)n}(B'_{\mathbf{q}_1}\cap B'_{\mathbf{q}_2}).$$

Proof. Consider the defining inequalities for each set on the left hand sides. Multiplying by X^{-1}, we obtain a new system of inequalities, so that

$$2^n\epsilon\prod_i\psi_i(|\mathbf{q}|) \leq \lambda_{(m-n)n}(B'_{\mathbf{q}}(X)) \leq 2^n\epsilon^{-1}\prod_i\psi_i(|\mathbf{q}|).$$

Considering the special case when $X = I_n$, we obtain the first statement.

The second statement is derived similarly, namely by considering the defining inequalities and multiplying by X^{-1} to get an estimate for the measure. □

Lemma 3.2. *For each pair $\mathbf{q},\mathbf{q}'$,*

$$\lambda_{mn}(B_{\mathbf{q}}) \asymp_\epsilon \lambda_{(m-n)n}(B'_{\mathbf{q}}), \tag{16}$$

and

$$\lambda_{mn}(B_{\mathbf{q}} \cap B_{\mathbf{q}'}) \asymp_\epsilon \lambda_{(m-n)n}(B'_{\mathbf{q}} \cap B'_{\mathbf{q}'}). \tag{17}$$

Proof. This follows on integrating out the X and applying Lemma 3.1. Indeed,

$$\lambda_{mn}(B_{\mathbf{q}}) \asymp_\epsilon \int_{\substack{X \in M_n([-1/2,1/2]) \\ \epsilon < |\det(X)|}} \int_{\tilde{A} \in M_{(m-n)\times n}([-1/2,1/2])X^{-1}} \mathbf{1}_{B_{\mathbf{q}}}\left(\begin{pmatrix} I_n \\ \tilde{A} \end{pmatrix} X\right) d\tilde{A} dX,$$

where $\mathbf{1}_{B_{\mathbf{q}}}$ denotes the characteristic function of $B_{\mathbf{q}}$. Let us prove that the inner integral is $\asymp_\epsilon \lambda_{(m-n)n}(B'_q(X))$.

First, we deal with the case when $m - n > 1$. For simplicity, we consider first the case $m = 3, n = 1$ and extend subsequently. We are integrating over the set $M_{2\times 1}([-1/2, 1/2])X^{-1}$, which is a square of area between 1 and ϵ^{-2}, since X in this case is just a number between ϵ and 1. Consider the intersection with each fundamental domain for the standard lattice $\mathbb{Z}^2$. Except for lower order terms arising at the boundary of $M_{2\times 1}([-1/2, 1/2])X^{-1}$, each such intersection will have measure $\lambda_{(m-n)n}(B'_{\mathbf{q}}(X))$. The number of such contributing fundamental domains is bounded from below by 1 and from above by ϵ^{-2}. Hence, the result follows in this case.

To get the full result for $m - n > 1$, the set $M_{(m-n)\times n}([-1/2, 1/2])X^{-1}$ still covers at least $\frac{1}{n}M_{(m-n)\times n}([-1/2, 1/2])$, as the entries of X are between $-1/2$ and $1/2$. For $|\mathbf{q}|$ large enough, the measure of the intersection of $B'_{\mathbf{q}}(X)$ with this set is $\asymp \frac{1}{n^{(m-n)n}}\lambda_{(m-n)n}(B'_{\mathbf{q}}(X))$, and the result follows. The upper bound again follows as the determinant of X is bounded from below.

When $m - n = 1$, the set consists of neighbourhoods of single points, and we simply count the contributions as usual. We have now shown that the inner integral is $\asymp_\epsilon \lambda_{(m-n)n}(B'_q(X))$.

To conclude, we use Lemma 3.1,

$$\begin{aligned} \lambda_{mn}(B_{\mathbf{q}}) &\asymp_\epsilon \int_{\substack{X \in M_n([-1/2,1/2]) \\ \epsilon < |\det(X)|}} \lambda_{(m-n)n}(B'_q(X)) dX \\ &\asymp_\epsilon \int_{\substack{X \in M_n([-1/2,1/2]) \\ \epsilon < |\det(X)|}} \lambda_{(m-n)n}(B'_{\mathbf{q}}) dX \asymp_\epsilon \lambda_{(m-n)n}(B'_{\mathbf{q}}). \end{aligned}$$

The case of intersections follows similarly, this time using the second equation of Lemma 3.1. ☐

At this point, proving the divergence case of the theorem is a relatively straightforward matter. Indeed, a form of the divergence case of the Borel–Cantelli lemma states that if (A_n) is a sequence of sets in a probability space with probability measure μ such that $\sum \mu(A_n) = \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) \geq \limsup_{N\to\infty} \frac{\left(\sum_{n=1}^N \mu(A_n)\right)^2}{\sum_{m,n=1}^N \mu(A_m \cap A_n)}. \tag{18}$$

If one can prove that for a sufficiently large set of pairs (A_n, A_m), the denominator on the right hand side is $\ll \mu(A_m)\mu(A_n)$ whenever $m \neq n$, it follows from (18) that the measure of the set on the left hand side is strictly positive. Even if this does not hold, one could hope for it to be true on average, so that the resulting right hand side would be positive. This is a standard technique in metric Diophantine approximation, with the property on the sets A_n being called quasi-independence or

in the latter case quasi-independence on average. It follows from Lemma 3.2, that if a classical Khintchine–Groshev theorem can be established using quasi-independence on average, then the measure of the absolute value set is positive under the appropriate divergence assumption.

In the classical setup, one usually proves Khintchine–Groshev type results using a variant of this lemma. Here, one applies the lemma with some subset of the family $B'_{\mathbf{q}}$ in place of A_n. In the simplest case, when $n = 1$ and $m = 3$, the family can be chosen to be those $\mathbf{q} = (p, \tilde{\mathbf{q}}) \in \mathbb{Z} \times \mathbb{Z}^2$ with the entries of $\tilde{\mathbf{q}}$ co-prime and the last entry positive. This will ensure that the corresponding sets $\cup_p B'_{(p,\tilde{\mathbf{q}})}$ are stochastically independent and hence quasi-independent. The fact that we take a union over p's s critical. This gives a pleasing description of the sets involved as neighbourhoods of geodesics winding around a torus, and provides a simple argument for the stochastic independence of the sets. For details on this case, see [7]. In that paper, the case $m - n > 1$ is fully described. For the case when $m - n = 1$, more delicate arguments are required. Below, we give references to work, where the refining procedure is carried out in each individual case.

For our purposes, in order to prove Theorem 1.1, using Lemma 3.2 we will translate the right hand side of inequality (18) to a statement on the 'classical' sets $B'_{\mathbf{q}}$ with the corresponding limsup set. In the case $m - n > 2$, the required upper bound on the intersections on average was established in [20] without the monotonicity assumption. For $m - n = 2$, the bound is found in [16] and $m - n = 1$, this is the result of [12]. In the last case, the monotonicity is critical in the case $m = 2, n = 1$, as otherwise we could exploit the Duffin–Schaeffer counterexample [8] to arrive at a counterexample to the present statement.

Having established that the measure is positive, it remains to prove that the measure is full. To accomplish this, we apply an inflation argument due to Cassels [3], but tweaked to the absolute value setup. We pick a slowly decreasing function $\tau(r)$ which tends to 0, such that the functions $\psi'_i(r) = \tau(r)\psi_i(r)$ satisfy the divergence assumption of the theorem.

One can show that the origin $0 \in \mathrm{Mat}_{mn}(\mathbb{R})$ is a point of metric density for the set $W_0(m, n; \underline{\psi})$. This uses two properties. One is the fact that 0 is an inner point of each set of matrices satisfying (1) for a fixed $\mathbf{q}$. The other is the fact that the error function depends only on $|\mathbf{q}|$, so the parallelepiped of matrices satisfying (1) does not change shape but only orientation as $\mathbf{q}$ varies over integer vectors with the same height $|\mathbf{q}|$. Since the distribution of angles of integer vectors of the same height becomes uniform as the height increases, this implies that the origin must be a point of metric density.

Now, by the Lebesgue Density Theorem, for almost every matrix $A \in \mathbb{R}^{mn}$, there is a matrix near the origin $\tilde{A} \in W_0(m, n; \underline{\psi}')$ and a real number r, such that $A = r\tilde{A}$. That is,

$$|\mathbf{q}A|_i = \left|\mathbf{q}r\tilde{A}\right|_i = |r| \left|\mathbf{q}\tilde{A}\right|_i < r\psi'_i(|\mathbf{q}|),$$

for infinitely many $\mathbf{q}$. This implies that $A \in W_0(m, n; \underline{\psi})$, since r is fixed and τ tends to 0.

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